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Next to leading order calculation with dimensional regularization in Nambu–Jona-Lasinio Model

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The Nambu–Jona-Lasinio model is investigated in the $1/N_c$ expansion with the dimensional regularization. At the four-dimensional limit the meson propagators have simple forms in the leading order of the $1/N_c$ expansion. Thus the next to leading order calculation reduces to an ordinary one loop calculation. Here we obtain an explicit form of the $1/N_c$ correction and numerically evaluate the N_c dependence for the gap equation.

Keywords: low energy effective theory; dimensional regularization; $1/N_c$ expansion.

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1. Introduction

The strong interaction between quarks and gluons are described by quantum chromodynamics (QCD). Because of the asymptotic freedom, the non-perturbative effect is essential for low energy phenomena in QCD. Nambu and Jona-Lasinio introduce a four-fermion interaction to study the meson properties with the strong interaction.¹ The NJL model has a similar symmetry behavior to QCD. It is used as one of the effective models of QCD for low energy.^{2–4}

The critical behavior for quarks and gluons is often discussed at the leading order of $1/N_c$ expansion in the NJL model. The number of colors, N_c , is three in the real world. Thus the quantitative properties can be determined with an accuracy of about 30% in the leading order of $1/N_c$ expansion. There is another ambiguity for the results in the NJL model. Since the four-fermion interaction is irrelevant, it is necessary to regularize the model in four dimensions. The result depends on the regularization parameter.^{5–8} The parameter is usually fixed to be consistent with the light meson properties.

In our previous study Ref. [9], we considered the NJL model with the dimensional regularization in the four-dimensional limit. It is found that the calculations of meson mass and decay constant are simplified in the leading order of the $1/N_c$

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expansion. Then it is expected that the analysis may also be simple if we proceed to the next to leading order level.

In this letter, we discuss the next to leading order of the $1/N_c$ expansion, and study the possible N_c dependence in the model predictions. This is interesting because the N_c dependence is absorbed in the other parameters in the leading order, and it first appears from the next to leading order.

2. $1/N_c$ expansion in the NJL model

In this section we briefly review the $1/N_c$ expansion in the NJL model for up and down quarks. The Lagrangian of two-flavor NJL model is given as,

$$\mathcal{L} = \sum_{j=1}^{N_c} \bar{\psi}_j (i\partial\!\!\!/ - m) \psi_j + g \sum_{j=1}^{N_c} [(\bar{\psi}_j \psi_j)^2 + (\bar{\psi}_j i\gamma_5 \tau^a \psi_j)^2], \quad (1)$$

where m is the current quark mass, τ^a is the Pauli matrices in the flavor space, and N_c is the number of colors which is treated as one of model parameters in this paper. We neglect the mass difference between up and down quarks. In the scheme of the $1/N_c$ expansion we take the large N_c limit as $N_c g$ fixed.

By applying the auxiliary field method, we can evaluate the generating functional from Eq. (1) as

$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\sigma \mathcal{D}\pi \exp \left[i \int d^4x \left\{ \bar{\psi} (i\partial\!\!\!/ - m) \psi - \bar{\psi} (\sigma + i\gamma_5 \tau^a \pi^a) \psi - \frac{1}{4g} [\sigma^2 + (\pi^a)^2] \right\} \right] \quad (2)$$

$$= \int \mathcal{D}\sigma \mathcal{D}\pi \exp [iI(\sigma, \pi)], \quad (3)$$

with

$$I(\sigma, \pi) = -\frac{1}{4g} \int d^4x \{ \sigma^2 + (\pi^a)^2 \} - i \ln \text{Det} (i\partial\!\!\!/ - m - \sigma - i\gamma_5 \tau^a \pi^a). \quad (4)$$

In Eq. (4) “Det” takes the color, flavor, spinor and space-time indices. $I(\sigma, \pi)$ can formally be expanded around the classical solution σ_0 and π_0 ,

$$I(\sigma, \pi) = I(\sigma_0, \pi_0) + \frac{1}{2} \frac{\delta^2 I(\sigma_0, \pi_0)}{\delta \sigma^2} (\sigma - \sigma_0)^2 + \frac{1}{2} \frac{\delta^2 I(\sigma_0, \pi_0)}{\delta \pi^2} (\pi - \pi_0)^2 + \dots \quad (5)$$

In the leading order of the expansion, the effective potential, $V = (i/\int d^4x) \ln Z$, is written by

$$V_0(\sigma, \pi) = \frac{1}{4g} \{ \sigma^2 + (\pi^a)^2 \} - \int \frac{d^4k}{i(2\pi)^4} \text{tr} \ln (\not{k} - m - \sigma - i\gamma_5 \tau^a \pi^a), \quad (6)$$

where “tr” takes the color, flavor and spinor indices.

From the stable condition $\partial V/\partial\sigma = 0$ at $\sigma = \sigma_0$, one can derive the gap equation whose leading order form becomes

$$\sigma_0 = 2N_f g \text{itr} S(m^*), \quad (7)$$

where $m^* = m + \sigma_0$ and

$$i \text{tr} S(m^*) = - \int \frac{d^4 k}{i(2\pi)^4} \text{tr} \frac{1}{\not{k} - m_i^* + i\varepsilon}. \quad (8)$$

“tr” in the integral denotes the trace with respect to the spinor and color indices.

In the leading order of the $1/N_c$ expansion, the propagator of pion is given by¹⁰

$$\Delta_\pi(p^2) = \frac{2g}{1 - 4g\Pi_5(p^2)}. \quad (9)$$

The pion mass is determined at the pole position of the propagator, namely,

$$1 - 4g\Pi_5(p^2 = m_\pi^2) = 0, \quad (10)$$

where

$$\begin{aligned} \Pi_5(p^2) &= - \int \frac{d^4 k}{i(2\pi)^4} \text{tr} [i\gamma_5 S(k) i\gamma_5 S(k-p)] \\ &= \frac{i \text{tr} S}{m^*} + \frac{1}{2} p^2 J(p^2), \end{aligned} \quad (11)$$

with

$$J(p^2) = \int \frac{d^4 k}{i(2\pi)^4} \text{tr} \frac{1}{(k^2 - m^{*2}) \{(k-p)^2 - m^{*2}\}}. \quad (12)$$

Similarly, the propagator of sigma meson is given as¹⁰

$$\Delta_\sigma(p^2) = \frac{2g}{1 - 4g\Pi_s(p^2)}, \quad (13)$$

where

$$\Pi_s(p^2) = - \int \frac{d^4 k}{i(2\pi)^4} \text{tr} [S(k) S(k-p)]. \quad (14)$$

The mass of the sigma meson is evaluated at the pole position as well.

3. Parameter fixing in the 4 dimensional limit

Since the model is not renormalizable, the predictions depend on the regularization procedure. Here we shall employ the dimensional regularization in the 4 dimensional limit.⁹ In this method, although the intermediate integrals diverge, the model predictions turn out to be finite thanks to the parameter fixing.

In our model treatment, we have three parameters, m , g and M . m is the current quark mass, g is the four point coupling and M is the mass rescaling parameter. In this paper we take m as the input parameter, the remaining two parameters will be fixed by using the physical observables. We try to fix these parameters with m_π and f_π in the leading order of the $1/N_c$ expansion.

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In the 4 dimensional limit, $D(\equiv 4 - 2\epsilon) \rightarrow 4$, the pion propagator can be written by¹⁰

$$\Delta_\pi(p^2) = -\frac{Z_\pi M^{2\epsilon}}{p^2 - m_\pi^2}. \quad (15)$$

where the wave function renormalization is

$$Z_\pi^{-1} = \frac{N_c}{4\pi^2\epsilon} M^{2\epsilon}. \quad (16)$$

Then the pion propagator becomes

$$\Delta_\pi(p^2) = -\frac{4\pi^2\epsilon}{N_c} \frac{1}{p^2 - m_\pi^2}. \quad (17)$$

The pion decay constant is also calculated in the same procedure,

$$f_\pi^2 = \frac{N_c}{4\pi^2\epsilon} M^{2\epsilon} \sigma_0^2. \quad (18)$$

Next, let us consider the gap equation. Performing the integral Eq. (8) in the 4 dimensional limit and inserting the result into Eq. (7), we obtain

$$\sigma_0 = -\frac{N_f N_c g}{2\pi^2\epsilon} m^{*3}. \quad (19)$$

Since the following relation is derived from Eq. (11)

$$\Pi_5(m_\pi^2) = \frac{N_c}{8\pi^2\epsilon} (m_\pi^2 - 2m^{*2}), \quad (20)$$

we get, with the help of Eqs. (10) and (19),

$$\sigma_0 = -m - \frac{m_\pi^2}{4m} \left\{ 1 + \sqrt{1 + \frac{8m^2}{m_\pi^2}} \right\}, \quad (21)$$

and

$$g = -\frac{\pi^2\epsilon}{N_c} \frac{\sigma_0}{m^{*3}}. \quad (22)$$

Equation (18) reads the relation

$$\lim_{\epsilon \rightarrow 0} M^{2\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{4\pi^2\epsilon}{N_c} \frac{f_\pi^2}{\sigma_0^2}. \quad (23)$$

It is interesting to note that the parameter M approaches to 0 in the 4 dimensional limit. This is the essentially important property of the mass rescaling parameter; we can control the divergent integrals by virtue of the adjustment of mass dimensions through M .⁹

Using Eqs. (7), (21) and (23), we obtain the chiral condensate,

$$\begin{aligned} \langle \bar{u}u \rangle_0 &= -M^{2\epsilon} (\text{itr} S) \\ &= \frac{f_\pi^2 m_\pi^4}{8m^3} \left\{ 1 - \sqrt{1 + \frac{8m^2}{m_\pi^2}} \right\} + O(1/N_c). \end{aligned} \quad (24)$$

Note that the expansion of Eq. (24) in powers of m leads the Gell-Mann–Oakes–Renner relation,¹² $\langle \bar{u}u \rangle_0 \simeq -f_\pi^2 m_\pi^2 / (2m)$.

In the similar manner, we have for the sigma meson propagator

$$\Delta_\sigma(p^2) = -\frac{4\pi^2\epsilon}{N_c} \frac{1}{p^2 - m_{\sigma 0}^2}, \quad (25)$$

where the sigma mass can be written

$$m_{\sigma 0}^2 = \frac{m_\pi^2}{2m^2} \left(m_\pi^2 + 6m^2 + m_\pi \sqrt{m_\pi^2 + 8m^2} \right). \quad (26)$$

For $m_\pi = 135\text{MeV}$ and $m = 5.0\text{MeV}$, we get $m_{\sigma 0} \simeq 3700\text{MeV}$. This value corresponds with the one obtained in Ref. [10], where the realistic value, $400-550\text{MeV}$,¹³ is found in different dimension around $D \simeq 2$. Thus, in the four dimensional limit with the leading order calculation we find the deviation on the value of m_σ . This deviation may be modified by considering the next to leading order of $1/N_c$ expansion, which will be discussed in the following.

4. Next to leading order of the $1/N_c$ expansion

We have presented the analysis in the 4 dimensional limit within the leading order of $1/N_c$ expansion. We shall carry on the calculations up to the next to leading order in this section.

4.1. Formula for the next to leading order

We consider next to leading order of the $1/N_c$ expansion for the effective potential by using the technique of auxiliary field method Ref. [11]. The second and third term of Eq. (5) are

$$\left. \frac{\delta^2 I}{\delta\sigma(x)\delta\sigma(y)} \right|_{\sigma=\sigma_0, \pi=\pi_0} = -\frac{1}{2g} \delta^4(x-y) + i \text{tr}[S(x, y; m^*)S(y, x; m^*)], \quad (27)$$

and

$$\begin{aligned} \left. \frac{\delta^2 I}{\delta\pi^a(x)\delta\pi^b(y)} \right|_{\sigma=\sigma_0, \pi=\pi_0} &= -\frac{1}{2g} \delta^{ab} \delta^4(x-y) \\ &\quad + i \text{tr}[i\gamma_5 \tau^a S(x, y; m^*) i\gamma_5 \tau^b S(y, x; m^*)]. \end{aligned} \quad (28)$$

After integrating σ and π with $\sigma - \sigma_0 \equiv \sigma$ and $\pi - \pi_0 \equiv \pi$ in Eq. (3), we obtain the effective potential at the next to leading order of the $1/N_c$ expansion,

$$\begin{aligned} V(\sigma, \pi) &= V_0(\sigma, \pi) \\ &\quad + \frac{1}{2} \int \frac{d^4 p}{i(2\pi)^4} \{ \ln(\Delta_\sigma^{-1}(p^2; m + \sigma)) + \ln(\Delta_\pi^{-1}(p^2; m + \sigma)) \}, \end{aligned} \quad (29)$$

here the relation of the minimum of the classical field σ and the classical solution σ_0 is $\sigma = \sigma_0 + O(1/N_c)$.¹¹

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From the stationary condition and taking $\pi = 0$, we have the gap equation until next to the leading order,

$$\langle \sigma \rangle = 2N_f g_{\text{itr}} S(m + \langle \sigma \rangle) - g \int \frac{d^4 p}{i(2\pi)^4} \frac{\partial}{\partial \sigma} \left\{ \ln(\Delta_\sigma^{-1}(p^2; m + \sigma)) + \ln(\Delta_\pi^{-1}(p^2; m + \sigma)) \right\} \Big|_{\sigma=\langle \sigma \rangle}. \quad (30)$$

Substituting the relation of the leading order of the $1/N_c$ expansion for Eq. (30), we obtain

$$\langle \sigma \rangle \simeq 2N_f g_{\text{itr}} S(m^*) - 2N_f g m^* \int \frac{d^D p}{i(2\pi)^D} \left(\frac{3}{m_{\sigma 0}^2 - p^2} + \frac{1}{m_\pi^2 - p^2} \right) \quad (31)$$

$$\simeq \sigma_0 \left(1 - \frac{\delta_\sigma}{N_c} \right), \quad (32)$$

where

$$\delta_\sigma = \frac{3m_{\sigma 0}^2 + m_\pi^2}{4m^{*2}}. \quad (33)$$

σ_0 denote the solution of the leading order of the $1/N_c$ expansion. By inserting the values for m_π , m^* and $m_{\sigma 0}$, we find $\delta_\sigma \simeq 3.0$.

Similarly, we can obtain the expressions for the chiral condensate and the sigma mass through a bit of algebra. It may be nice to summarize the equations in the next to leading order results which read

$$\langle \bar{u}u \rangle \simeq \langle \bar{u}u \rangle_0 \left[1 - \left(1 - 3 \frac{m}{m^*} \right) \frac{\delta_\sigma}{N_c} \right], \quad (34)$$

$$m_\sigma^2 = 2(\langle \sigma \rangle + m)^2 \left(2 - \frac{m}{\langle \sigma \rangle} \right) \quad (35)$$

$$\simeq m_{\sigma 0}^2 - 2\sigma_0^2 \left(4 + 3 \frac{m}{\sigma_0} + \frac{m^3}{\sigma_0^3} \right) \frac{\delta_\sigma}{N_c}, \quad (36)$$

these relations are derived from Eqs. (24) and (13).

4.2. Numerical results

Having obtained the expression for the next to leading order of the $1/N_c$ expansion, we are now ready for performing the numerical analysis.

Below we show the results for $\langle \sigma \rangle$, $\langle \bar{u}u \rangle^{1/3}$ and m_σ with respect to N_c in Figs. 1, 2 and 3. The model parameters are determined by the input values: $m = 5.0\text{MeV}$, $m_\pi = 135\text{MeV}$ and $f_\pi = 92\text{MeV}$, as mentioned above. The solid curves indicate the results for the next to leading order, and the dashed lines are the ones in the leading order. We see that $\langle \sigma \rangle$ and $\langle \bar{u}u \rangle$ decrease according to N_c , while m_σ increases when N_c becomes large. As trivially expected, the values approach to the ones in the leading order which are shown in dashed lines. It is interesting to note that the realistic values may be found in the region $3 < N_c < \infty$, then we think the expansions performed in the 4 dimensional limit effectively work.

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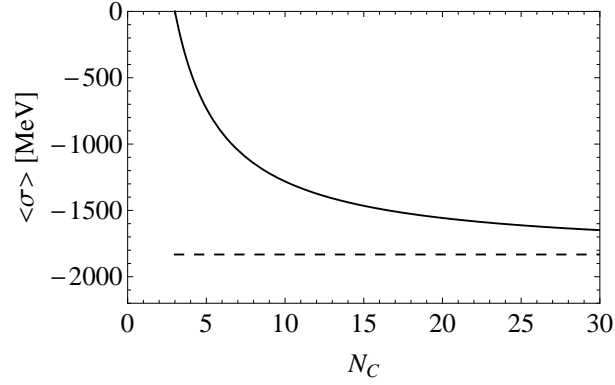


Fig. 1. N_c dependence on $\langle \sigma \rangle$. Dashed line: $\sigma_0 = -1832.5 \text{ MeV}$.

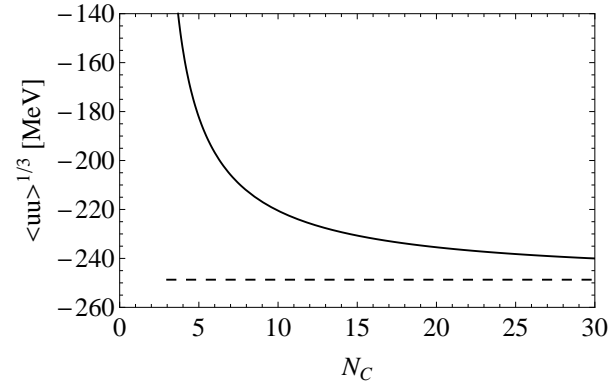


Fig. 2. N_c dependence on $\langle \bar{u}u \rangle^{1/3}$. Dashed line: $\langle \bar{u}u \rangle_0^{1/3} = -248.7 \text{ MeV}$.

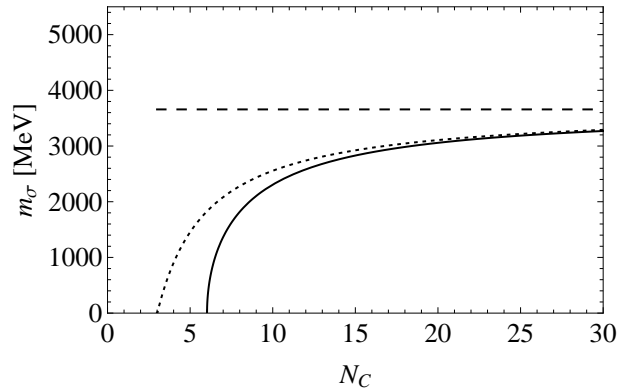


Fig. 3. N_c dependence on m_σ shown in the solid line (from Eq. (36)). Dashed line: $m_\sigma = 3657.5 \text{ MeV}$. Dotted curve is the result from the intermediate relation in Eq. (35).

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5. Concluding remarks

We have considered the $1/N_c$ correction by taking the four dimensional limit in the NJL model. We first perform the calculation up to the next to leading order of the $1/N_c$ expansions. There we found that the calculations are drastically simplified thanks to the manipulation of taking the four dimensional limit, and we are able to study the N_c dependence in the systematic manner. We also check the numerical tendencies with respect to N_c on various physical quantities, then show the predictions of them in which the values approach to that of the leading order case for large N_c .

One important qualitative advantage in this analysis is that we can perform the next to leading order calculations in an easy way. While there is quantitative unsatisfactory point on the value of m_σ being considerably larger than the observed one, as previously found in the corresponding study in which the analysis is restricted to the leading order case.¹⁰ However, the result indicates that the discrepancy may be cured through including the higher order corrections since the mass of sigma becomes smaller if we consider the next to leading order contribution. Therefore, we think further investigations on the expansion of N_c are interesting and important.

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